STABILITY OF A VISCOELASTIC INHOMOGENEOUS SHELL

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Shells are considered for which the constants of viscoelasticity are random functions of curvilinear coordinates of the middle surface. Correlation functions are obtained for the first approximations of the deflection and the stress function, as well as the variance of the critical time.

A peculiarity of shells subjected to compression under the conditions of unlimited creep is their "snapping" at any load after the elapse of a longer or shorter time interval, called the critical time. Its magnitude depends on many factors, and in the first instance, on the characteristics of the elastic and viscous properties of the material. The latter, as is known [1], have a considerable scatter, being random functions of the coordinates. As a result, the shell is nonhomogeneous, and this leads to redistribution of the forces in the middle surface with time, and as a consequence, to variation in the critical time.

The stability of homogeneous shells in creep, in a geometrically nonlinear formulation, was considered in [2, 3].

We consider a viscoelastic thin shell the properties of whose material are described by random functions of curvilinear coordinates of the middle surface.

Assuming, for the sake of simplicity, the material to be incompressible, we write the relationships between the strains and stresses

$$\varepsilon_{ij} = \frac{1}{2G}(1+K)s_{ij} \qquad (i, j=1, 2)$$

where

$$\frac{1}{G}Ks_{ij} = 3A\int_0^t s_{ij}(\tau)\,d\tau, \qquad s_{ij} = \sigma_{ij} - \delta_{ij}\sigma, \qquad 3\sigma = \sigma_{ii}$$

G is the shear modulus, A is a constant characterizing the viscosity, t and τ are time, and δ_{ij} is a unit tensor.

Here and in the following summation is carried out over repeated indices. The index numbers correspond to the coordinates x_1 and x_2 which are measured along the lines of curvature of the middle surface.

If the inhomogeneity is small,

$$1/G = \lambda = \langle\lambda\rangle + \beta\lambda', \;\; A = \langle A\rangle + \beta A', \; \langle\lambda\rangle, \; \langle A\rangle = \mathrm{const}$$

(β is a small parameter), the deflection w and the stress functions Φ can be found by the method of a small parameter in the form of a series

$$w = \sum_{k=0}^{\infty}\beta^k w^{(k)}, \qquad \Phi = \sum_{k=0}^{\infty}\beta^k\Phi^{(k)} \tag{1}$$

Angle brackets are used to note averaging over a set of realizations.

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We assume that the middle surface of the shell has small initial deviations from the ideal form $w_0 = \beta w_0$. Confining ourselves to the first approximation in the expansion (1), we write the equations determining $w^{(1)}$ and $\Phi^{(1)}$ in the case of a thin shallow shell:

$$D\nabla^4 (w^{(1)} - w_0) = (1 + \langle K \rangle) \left[(\nabla^2 \Phi^{(1)} \delta_{ij} - \Phi^{(1)}_{,ij}) \frac{1}{R_{ij}} + (\nabla^2 \Phi^{(0)} \delta_{ij} - \Phi^{(0)}_{,ij}) w^{(1)}_{,ij} \right]$$

$$2\langle \lambda \rangle (1 + \langle K \rangle) \nabla^4 \Phi^{(1)} + \frac{6h}{R_{22}} (w^{(1)} - w_0)_{,11} + \tag{2}$$
$$+ \frac{6h}{R_{11}} (w^{(1)} - w_0)_{,22} = -[\lambda (1 + K)(2\Phi^{(0)}_{,22} - \Phi^{(0)}_{,11})]_{,22} -$$
$$- [\lambda (1 + K)(-\Phi^{(0)}_{,22} + 2\Phi^{(0)}_{,11})]_{,11} - 6 [\lambda (1 + K) \Phi^{(0)}_{,12}]_{,12}$$

where

$$\langle K \rangle \nabla^4 \Phi^{(1)} = 3 \frac{\langle A \rangle}{\langle \lambda \rangle} \int_0^t \nabla^4 \Phi^{(1)} (\tau)\, d\tau, \qquad I = h^3/12, \qquad D = 4I/\langle \lambda \rangle$$

R_{11} and R_{22} are the radii of the curvatures $(1/R_{12} = 1/R_{21} = 0)$.

For a homogeneous momentless state (for $k = 0$ $\Phi^{(0)}_{,ij} = \mathrm{const}$) the second one of these relationships is simplified,

$$2\langle \lambda \rangle (1 + \langle K \rangle) \nabla^4 \Phi^{(1)} + \frac{6h}{R_{22}} (w^{(1)} - w_0)_{,11} +$$
$$+ \frac{6h}{R_{11}} (w^{(1)} - w_0)_{,22} = -(\lambda + \lambda K)_{,22} (2\Phi^{(0)}_{,22} - \Phi^{(0)}_{,11}) -$$
$$- (\lambda + \lambda K)_{,11} (-\Phi^{(0)}_{,22} + 2\Phi^{(0)}_{,11}) - 6 (\lambda + \lambda K)_{,12} \Phi^{(0)}_{,12} \tag{3}$$

We assume that the scales vary and the correlation functions λ' and A' are small in comparison with the characteristic dimensions of the middle surface, and that the functions themselves are homogeneous. Then they can be represented by the stochastic Fourier–Stieltjes integrals [4]

$$\lambda'(x) = \int_{-\infty}^{\infty} e^{i\omega x} dZ_\lambda(\omega), \quad A'(x) = \int_{-\infty}^{\infty} e^{i\omega x} dZ_A(\omega), \quad \omega x = \omega_1 x_1 + \omega_2 x_2$$

The functions $Z_\lambda(\omega)$ and $Z_A(\omega)$ satisfy the conditions

$$\langle dZ_\lambda(\omega)\, dZ_\lambda^*(\omega') \rangle = S_\lambda(\omega)\, \delta(\omega - \omega')\, d\omega d\omega'$$
$$\langle dZ_A(\omega)\, dZ_A^*(\omega') \rangle = S_A(\omega)\, \delta(\omega - \omega')\, d\omega d\omega'$$

Here $\delta(\omega - \omega')$ is a two-dimensional delta function; $S_\lambda(\omega)$ and $S_A(\omega)$ are the spectral densities of the random functions $\lambda'(x)$ and $A'(x)$. An asterisk is used to denote transition to complex conjugate quantities.

If the fields of $\lambda'(x)$, $A'(x)$ are not only homogeneous but also homogeneously connected, then

$$\langle dZ_\lambda(\omega)\, dZ_A^*(\omega') \rangle = S_{\lambda A}(\omega)\, \delta(\omega - \omega')\, d\omega d\omega'$$

The solution of Eqs. (2) and (3) is given by the expressions

$$w^{(1)} = v(x, t) + \int_{-\infty}^{\infty} e^{i\omega x} \psi_A(\omega)\, dZ_A(\omega) + \int_{-\infty}^{\infty} e^{i\omega x} \psi_\lambda(\omega)\, dZ_\lambda(\omega)$$
$$\Phi^{(1)} = f(x, t) + \int_{-\infty}^{\infty} e^{i\omega x} \varphi_A(\omega)\, dZ_A(\omega) + \int_{-\infty}^{\infty} e^{i\omega x} \varphi_\lambda(\omega)\, dZ_\lambda(\omega) \tag{4}$$

where

$$\psi_A(\omega) = \frac{B(\omega)}{2\langle A \rangle \gamma(\omega) a(\omega)} \left(\frac{\omega_1^2}{R_{22}} + \frac{\omega_2^2}{R_{11}} \right) \left\{ \exp\left[-\frac{3\langle A \rangle \gamma(\omega)}{\langle \lambda \rangle} t \right] - 1 \right\}$$
$$\psi_\lambda(\omega) = \frac{B(\omega)}{2\langle \lambda \rangle a(\omega)} \left(\frac{\omega_1^2}{R_{22}} + \frac{\omega_2^2}{R_{11}} \right) \exp\left[-\frac{3\langle A \rangle \gamma(\omega)}{\langle \lambda \rangle} t \right]$$
$$\varphi_A(\omega) = \frac{B(\omega)}{2\langle A \rangle (\omega_1^2 + \omega_2^2)^2} \left\{ 1 - \mu(\omega) \exp\left[-\frac{3\langle A \rangle \gamma(\omega)}{\langle \lambda \rangle} t \right] - [1 - \mu(\omega)] \exp\left(-\frac{3\langle A \rangle}{\langle \lambda \rangle} t \right) \right\}$$
$$\varphi_\lambda(\omega) = \frac{B(\omega)}{2\langle \lambda \rangle (\omega_1^2 + \omega_2^2)^2} \left\{ [1 - \mu(\omega)] \exp\left(-\frac{3\langle A \rangle}{\langle \lambda \rangle} t \right) + \gamma(\omega) \mu(\omega) \exp\left[-\frac{3\langle A \rangle \gamma(\omega)}{\langle \lambda \rangle} t \right] \right\}$$
$$B(\omega) = \omega_2^2 (2\Phi^{(0)}_{,22} - \Phi^{(0)}_{,11}) + \omega_1^2 (-\Phi^{(0)}_{,22} + 2\Phi^{(0)}_{,11}) + 6\omega_1 \omega_2 \Phi^{(0)}_{,12}$$
$$a(\omega) = D(\omega_1^2 + \omega_2^2)^4 + \frac{3h}{\langle \lambda \rangle} \left(\frac{\omega_1^2}{R_{22}} + \frac{\omega_2^2}{R_{11}} \right)^2 + (\omega_1^2 + \omega_2^2)^2 \omega_i \omega_j (\nabla^2 \Phi^{(0)} \delta_{ij} - \Phi^{(0)}_{,ij})$$

$$\gamma(\omega) = (\omega_1^2 + \omega_2^2)^2 \omega_i \omega_j (\nabla^2 \Phi^{(0)} \delta_{ij} - \Phi^{(0)}_{,ij}) a^{-1}(\omega)$$

$$\mu(\omega) = \frac{3h}{\langle\lambda\rangle [1 - \gamma(\omega)] a(\omega)} \left(\frac{\omega_1^2}{R_{22}} + \frac{\omega_2^2}{R_{11}} \right)^2$$

where v(x, t), f(x, t) are the deflection and the stress functions which correspond to the initial deflection w_0.

The relationships just presented have meaning for values of q and t which are less than the critical values for the viscoelastic constants $<\lambda>$ and $<A>$.

The correlation functions of flexure and stress functions, $K_W(x, x')$ and $K_\Phi(x, x')$, have the form

$$K_w(x, x') = \mathrm{Re} \int_{-\infty}^{\infty} e^{i\omega(x-x')} S_w(\omega)\, d\omega, \quad K_\Phi(x, x') = \mathrm{Re} \int_{-\infty}^{\infty} e^{i\omega(x-x')} S_\Phi(\omega)\, d\omega,$$

where $S_W(\omega)$, $S_\Phi(\omega)$ are the spectral densities of the functions w(x), Φ(x).

$$S_w(\omega) = \psi_A^2(\omega) S_A(\omega) + \psi_A(\omega) \psi_\lambda(\omega) [S_{A\lambda}(\omega) + S_{\lambda A}(\omega)] + \psi_\lambda^2(\omega) S_\lambda(\omega)$$

$$S_\Phi(\omega) = \varphi_A^2(\omega) S_A(\omega) + \varphi_A(\omega) \varphi_\lambda(\omega) [S_{A\lambda}(\omega) + S_{\lambda A}(\omega)] + \varphi_\lambda^2(\omega) S_\lambda(\omega)$$

To determine the critical time, we consider the perturbed motion of the shell. The perturbation δw of the deflection and $\delta\Phi$ of the stress function are found from the equations

$$\begin{aligned} &4I [G \nabla^4 \delta w^{\cdot} + 2G_{,i} \nabla^2 \delta \dot{w}_{,i} + G_{,ij} (\delta \dot{w}_{,ij} + \delta_{ij} \nabla^2 \delta w^{\cdot})] - \\ &- (\nabla^2 \Phi \delta_{ij} - \Phi_{,ij}) \delta \dot{w}_{,ij} - \left(w_{,ij} + \frac{1}{R_{jj}} \right) (\nabla^2 \delta \Phi^{\cdot} \delta_{ij} - \delta \dot{\Phi}_{,ij}) = \ldots \\ &2\lambda \nabla^4 \delta \Phi^{\cdot} + 4\lambda_{,i} \nabla^2 \delta \dot{\Phi}_{,i} + \lambda_{,ij} (3\delta \dot{\Phi}_{,ij} - \nabla^2 \delta \Phi^{\cdot} \delta_{ij}) + 6h \left(\frac{1}{R_{11}} \delta \dot{w}_{,22} + \right. \\ &\left. + \frac{1}{R_{22}} \delta \dot{w}_{,11} + w_{,11} \delta \dot{w}_{,22} + w_{,22} \delta \dot{w}_{,11} - 2w_{,12} \delta \dot{w}_{,12} \right) = \ldots \end{aligned} \quad (5)$$

In the right sides of these equations we have quantities which do not depend on $\delta w^{\cdot}$ and $\delta\Phi^{\cdot}$.

The critical time is obtained from the condition that the velocities $\delta w^{\cdot}$ and $\delta\Phi^{\cdot}$ increase without bounds [5]. This in the given case gives the same result as the bifurcation criterion of the equilibrium position [2].

As an example, we calculate the probabilistic characteristics of the critical time for a cylindrical shell of radius R, compressed along the generator by a load q. Let the elastic constant λ (or G) be deterministic, while the viscosity parameter A be a homogeneous random function only of the x_1 coordinate (x_1 is measured along the generator). The shell has an initial deflection $w_0 = v_0 \sin^2 \pi m x_1 / l$ (l is the length of the shell).

The expressions (4) in this case assume the form

$$w^{(1)}(x_1) = v \sin^2 \frac{m\pi}{l} x_1 + \frac{1}{2\langle A\rangle R} \int_{-\infty}^{\infty} e^{i\omega_1 x_1} \frac{1}{\omega_1^2} \left[1 - \exp\left(\frac{E\langle A\rangle \alpha_\omega}{1 - \alpha_\omega} t \right) \right] dZ_A(w_1)$$

$$\Phi^{(1)}(x_1) = f \sin^2 \frac{m\pi}{l} x_1 + \frac{1}{2\langle A\rangle} \int_{-\infty}^{\infty} e^{i\omega_1 x_1} \left\{ \frac{q}{\omega_1^2} (1 - e^{-E\langle A\rangle t}) - \frac{Eh\alpha_\omega}{R^2 \omega_1^4} \left[\exp\left(\frac{E\langle A\rangle}{1 - \alpha_m} \alpha_m t \right) - e^{-E\langle A\rangle t} \right] \right\} dZ_A(\omega_1)$$

where

$$v = \frac{v_0}{1 - \alpha_m} \exp\left(\frac{E\langle A\rangle}{1 - \alpha_m} \alpha_m t \right), \qquad f = \frac{l^2 h E \alpha_m}{4m^2 \pi^2 R} v$$

$$\alpha_m = q \left[\frac{4m^2 \pi^2}{l^2} D + \frac{l^2 h E}{4m^2 \pi^2 R^2} \right]^{-1}, \quad \alpha_\omega = q \left[D\omega_1^2 + \frac{Eh}{R^2 \omega_1^2} \right]^{-1}, \quad E = 3G$$

To obtain the equations for the critical time t_*, we specify the functions δw and $\delta\Phi$ (the boundary conditions are satisfied in the mean)

$$\delta w = c \sin \frac{m\pi}{l} x_1 \sin \frac{n}{R} x_2, \quad \delta\Phi = d \sin \frac{m\pi}{l} x_1 \sin \frac{n}{R} x_2$$

Eqs. (5) are solved by the Galerkin–Kantorovich method, assuming $w \approx w^{(1)}$, $\Phi \approx \Phi^{(0)} + \Phi^{(1)}$. The critical time is found from the condition that the determinant

$$\delta - \rho v = \Delta$$

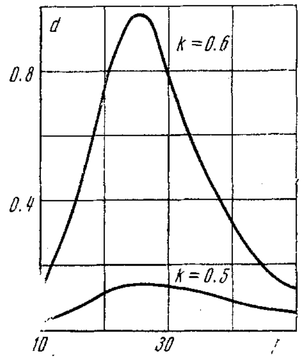

Fig. 1

consisting of the coefficients of c' and d, where

$$\delta = D\left(\frac{m^2\pi^2}{l^2} + \frac{n^2}{R^2}\right)^2 + \frac{m^4\pi^4 Eh}{R^2 l^4}\left(\frac{m^2\pi^2}{l^2} + \frac{n^2}{R^2}\right)^{-2} - \frac{m^2\pi^2}{l^2} q$$

$$\rho = \frac{n^2 Eh}{4R^3}\left[\alpha_m + 8\left(1 + \frac{n^2 l^2}{m^2\pi^2 R^2}\right)^{-2}\right]$$

$$\Delta = \frac{n^2 Eh}{2R^2 l}\int_{-\infty}^{\infty} \frac{\sin\omega_1 l + i(1-\cos\omega_1 l)}{\omega_1}\left(1 - \frac{l^2\omega_1^2}{4m^2\pi^2}\right)^{-1} \times$$

$$\times \left\{\frac{q}{Eh}\left(1 - e^{-E\langle A\rangle t}\right) + \frac{\alpha_\omega}{R^2\omega_1^2}\left[e^{-E\langle A\rangle t} - \exp\left(\frac{E\langle A\rangle}{1-\alpha_\omega}\alpha_\omega t\right)\right] + \right.$$

$$\left. + \frac{2l^2}{m^2\pi^2R^2}\left(1 + \frac{n^2l^2}{m^2\pi^2R^2}\right)^{-2}\left[1 - \exp\left(\frac{E\langle A\rangle}{1-\alpha_\omega}\alpha_\omega t\right)\right]\right\}\frac{1}{\langle A\rangle} dZ_A(\omega_1)$$

becomes zero.

Here we have omitted the terms which contain products of small quantities.

In the first approximation the critical time can be represented as the sum $t_* = t_0 + t_1$. Each of the terms is found from the equations

$$\delta = \rho v(t_0), \qquad t_1 = -\frac{1-\alpha_m}{\alpha_m E\langle A\rangle \delta}\Delta(t_0)$$

It is obvious that $\langle t_*\rangle \approx t_0$, while the variance $D(t_1)$ equals

$$D(t_1) = \frac{1}{2}\left[\frac{(1-\alpha_m)n^2 h}{\alpha_m\langle A\rangle R^2\delta}\right]^2 \int_{-\infty}^{\infty} \frac{1-\cos\omega_1 l}{l^2\omega_1^2(1 - l^2\omega_1^2/4m^2\pi^2)^2} \times$$

$$\times \left\{\frac{q}{Eh}\left(1 - e^{-E\langle A\rangle t}\right) + \frac{\alpha_\omega}{R^2\omega_1^2}\left[e^{-E\langle A\rangle t} - \exp\left(\frac{E\langle A\rangle}{1-\alpha_\omega}\alpha_\omega t\right)\right] + \right.$$

$$\left. + \frac{2l^2}{m^2\pi^2R^2}\left(1 + \frac{n^2l^2}{m^2\pi^2R^2}\right)^{-2}\left[1 - \exp\left(\frac{E\langle A\rangle}{1-\alpha_\omega}\alpha_\omega t\right)\right]^2 \frac{S_A(\omega_1)}{\langle A\rangle^2}\right\} d\omega_1 \tag{6}$$

We take the correlation function of the random function $A'(x_1)$ in the form

$$K_A(x_1 - x_1') = Q^2 e^{-s^2(x_1 - x_1')^2}$$

The spectral density for it is represented in the following manner:

$$S(\omega_1) = \frac{Q^2}{2\sqrt{\pi}s} e^{-\omega_1^2/4s^2}$$

We rewrite the expression (6) for the small quantities $E\langle A\rangle t$, $E\langle A\rangle(1-\alpha_\omega)^{-1}\alpha_\omega t$

$$D(t_1) = \frac{\sqrt{\pi}}{ls}\left(\frac{Q}{\langle A\rangle}\frac{t_0}{2\pi}\right)^2 \int_{-\infty}^{\infty} \frac{\theta^2(1-\cos 2\pi\theta)}{(1-\theta^2/m^2)^2} \times \frac{[\theta^2 - 3p(k+2g)]^2}{(\theta^4 - 3pk\theta^2 + p^2)^2}\exp\left(-\frac{\pi^2\theta^2}{l^2s^2}\right) d\theta \tag{7}$$

Here

$$\theta = \frac{l\omega_1}{2\pi}, \quad p = \frac{3l^2}{4\pi^2 Rh}, \quad \frac{q}{Eh} = k\frac{h}{R}$$

$$g = \frac{l^2}{m^2\pi^2 Rh}\left(1 + \frac{n^2l^2}{m^2\pi^2R^2}\right)^{-2}$$

When deriving Eq. (7) we assumed that the condition

$$nl / m\pi R = 1$$

was fulfilled.

We put $g = 1/3$. This value corresponds to the minimum value of the critical load for the shell in the linear formulation.

The results of the calculation of the variance $D(t_1)$ for $m = 5$, $ls = 20$ are shown in Fig. 1 as a function of the quantity p for various k. As is seen from the figure in which

$$d = \langle A\rangle^2 D(t_1) / Q^2 t_0^2$$

the relative deviation of the critical time $D(t_1)/t_0$ can be of the same order as $Q/\langle A \rangle$, and to a large extent depends on the value of the acting compressive load.

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